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1998 J. Phys. A: Math. Gen. 31 9283

(http://iopscience.iop.org/0305-4470/31/46/018)

J. Phys. A: Math. Gen. **31** (1998) 9283–9295. Printed in the UK PII: S0305-4470(98)94511-5

The determinant representation for quantum correlation functions of the sinh–Gordon model

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Received 21 May 1998

Abstract. We consider the quantum sinh–Gordon model in this paper. Form factors in this model were calculated by Mussardo and colleagues. We sum up all contributions of form factors and obtain a closed expression for a correlation function. This expression is a determinant of an integral operator. Similar determinant representations have been proven to be useful not only in the theory of correlation functions, but also in matrix models.

1. Introduction

The theory of massive, relativistic, integrable models is an important part of modern quantum field theory [18–25]. Scattering matrices in these models factorize into a product of two-body S-matrices [18]. Form factors can be calculated on the basis of a bootstrap approach [18–25].

The purpose of this paper is to calculate correlation functions. As usual correlation functions can be represented as an infinite series of form factor contributions. In this paper we sum up all these contributions and obtain a closed expression for correlation functions of local operators (4.8). The idea of this summation is the following. We introduce an auxiliary Fock space and auxiliary Bose fields (we shall call them dual fields). These fields help us to represent the form factor decomposition of a correlation function in a form similar to the 'free fermionic' case. This approach was developed in [6, 26, 27]. Finally, a correlation function is represented as a vacuum mean value (in the auxiliary Fock space) of a determinant of an integral operator (5.1). This representation has proven to be useful [6, 8, 12]; it helps in the asymptotical analysis of quantum correlation functions. Among other things, this approach has helped in the calculation of the asymptotics of the time- and temperature-dependent correlation functions in the nonlinear Schrödinger equation [9].

In this paper we consider the sinh–Gordon model. This is the model of one (real) relativistic Bose field ϕ in two dimensions. The action is

$$S = \int_{-\infty}^{\infty} \mathrm{d}^2x \left[\frac{1}{2}(\partial_\mu\phi(x))^2 - \frac{m_0^2}{g^2}\cosh g\phi(x)\right]. \tag{1.1}$$

It is the simplest example of the affine Toda field theories [28] with Z_2 symmetry $\phi \to -\phi$. The model has only one massive particle. The two-body scattering matrix [19, 29] is given

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by the expression

$$S(\beta, B) = \frac{\tanh \frac{1}{2}(\beta - (\mathrm{i}\pi B/2))}{\tanh \frac{1}{2}(\beta + (\mathrm{i}\pi B/2))} \qquad B = \frac{2g^2}{8\pi + g^2}. \tag{1.2}$$

We consider a real g, which corresponds to a positive B. Later we shall use a variable

$$x = \mathrm{e}^{\beta} \tag{1.3}$$

instead of the rapidity β.

We use the representation for form factors found in [1–3] (another representation can be found in [23])

$$F_n(\beta_1, \ldots, \beta_n) = \langle 0|\mathcal{O}(0)|\beta_1, \ldots, \beta_n\rangle = H_n Q_n(s) \prod_{i>j}^{n} \frac{F_{\min}(\beta_{ij})}{x_i + x_j}. \tag{1.4}$$

Here $\beta_{ij} = \beta_i - \beta_j$. A function $F_{\min}(\beta)$ is holomorphic for real β

$$F_{\min}(\beta) = \mathcal{N}(B)\,\Xi(\beta) \tag{1.5}$$

where

$$\Xi(\beta) = \exp\left[8\int_0^{\infty} \frac{\mathrm{d}x}{x} \frac{\sinh(\frac{1}{4}xB)\sinh(\frac{1}{2}x(1-\frac{1}{2}B))\sinh\frac{1}{2}x}{\sinh^2 x} \sin^2\left(\frac{x\hat{\beta}}{2\pi}\right)\right] \tag{1.6}$$

$$\mathcal{N}(B) = \exp\left[-4\int_0^{\infty} \frac{\mathrm{d}x}{x} \frac{\sinh(\frac{1}{4}xB)\sinh(\frac{1}{2}x(1-\frac{1}{2}B))\sinh\frac{1}{2}x}{\sinh^2 x}\right] \tag{1.7}$$

and $\hat{\beta} = \mathrm{i}\pi - \beta$. The function $F_{\min}(\beta)$ has a simple zero at $\beta = 0$ and no poles at the strip $0 \leqslant \mathrm{Im}\,\beta \leqslant \pi$. At $\beta \to \infty$ it goes to one: $F_{\min}(\beta) \to 1$. The functions $Q_n(x_1, \ldots, x_n)$ are symmetric polynomials of variables $x_1, \ldots, x_n$ given by [1]

$$Q_n(s) = \det{}_{n-1} M_{ij}(s) \qquad i, j = 1, \ldots, n-1 \tag{1.8}$$

where

$$M_{ij}(s) = \sigma_{2i-j}[i - j + s]. \tag{1.9}$$

Let us explain the notation. Here and later we suppress the dependency of $Q_n(s)$ on the variables x_j. The index $n-1$ in the expression $\det_{n-1}$ denotes the dimension of the matrix $M_{ij}(s)$. The functions σ_k are elementary symmetric polynomials of kth order in the variables $x_1, \ldots, x_n$:

$$\sigma_k \equiv \sigma_k^{(n)}(x_1, \ldots, x_n) = \sum_{i_1<i_2<\cdots<i_k}^{n} x_{i_1} x_{i_2} \ldots x_{i_k} \tag{1.10}$$

and $\sigma_k = 0$ if $k < 0$ or $k > n$. Here we have also suppressed the dependency of σ_k on x_j. The symbol $[m]$ is a 'q-number' defined by

$$[m] = \frac{\sin(mB\pi/2)}{\sin(B\pi/2)} = \frac{q^m - q^{-m}}{q - q^{-1}} \tag{1.11}$$

where $q = \exp\{\mathrm{i}B\pi/2\}$. The number s in (1.8) and (1.9) is an arbitrary integer, depending on the specific choice of the operator $\mathcal{O}$ in (1.4).

Finally, the constants H_k in (1.4) are normalization constants

$$H_{2n+1} = H_1\mu^n \qquad H_{2n} = H_0\mu^n \qquad \mu = \frac{4\sin(\pi B/2)}{F_{\min}(\mathrm{i}\pi)} \tag{1.12}$$

where H_0 and H_1 also depend on the specific operator $\mathcal{O}$. For instance, the form factor of the local field is given by (1.4) with $s = 0$ and

$$H_0 = \langle 0|\phi(0)|0\rangle = 0 \qquad H_1 = \langle 0|\phi(0)|\beta\rangle = \frac{1}{\sqrt{2}}. \tag{1.13}$$

A correlation function of an operator $\mathcal{O}$ can be presented as an infinite series of form factor contributions

$$\langle 0|\mathcal{O}(0,0)\mathcal{O}(x,t)|0\rangle = \sum_{n=0}^{\infty} \int \frac{\mathrm{d}^n\beta}{n!(2\pi)^n} |F_n(\beta_1,\ldots,\beta_n)|^2 \prod_{j=1}^{n} \mathrm{e}^{-mr\cosh\beta_j}. \tag{1.14}$$

In this paper we sum up this series explicitly. Section 2 is devoted to a transformation of the determinants (1.8) and (1.9) to a form which is convenient for summation. In section 3 we introduce auxiliary quantum operators—dual fields—in order to factorize an expression for a correlation function and to represent it in a form similar to the 'free fermionic case'. In section 4 we sum up the series (1.14) into a Fredholm determinant. In section 5 we use the Fredholm determinant representation for derivation of an asymptotic behaviour for correlation functions.

2. A transformation of the form factor

A determinant of a linear integral operator $I + V$ can be written as

$$\det(I+V) = \sum_{n=0}^{\infty} \int \frac{\mathrm{d}x_1\ldots\mathrm{d}x_n}{n!} \det_n \begin{pmatrix} V(x_1,x_1) & \cdots & V(x_1,x_n) \\ V(x_2,x_1) & \cdots & V(x_2,x_n) \\ \cdot & \cdot & \cdot \\ V(x_n,x_1) & \cdots & V(x_n,x_n) \end{pmatrix}. \tag{2.1}$$

Thus, in order to obtain a determinant representation for correlation functions one needs to represent the form factor expansion (1.14) in the form (2.1). Determinants of the integral operators that we consider can also be called Fredholm determinants.

The form factors (1.4) are proportional to the polynomials $Q_n(s)$, which in turn are equal to the determinants of the $(n-1)\times(n-1)$ matrices (1.8)

$$Q_n(s) = \det_{n-1} M_{ij}(s). \tag{2.2}$$

The matrix $M_{ij}(s)$ consists of $(n-1)^2$ different functions, depending on the same set of arguments $x_1,\ldots,x_n$:

$$M_{ij}(s) = \sigma_{2i-j}[i-j+s] \qquad i,j = 1,\ldots,n-1. \tag{2.3}$$

The main goal of this and the next section is to transform the matrix (2.3) to such a form, so that entries of a new matrix will be parametrized by a single function, depending on different sets of variables (such as $V(x_i,x_j)$ in (2.1))

$$M_{ij} \to \hat{D}_{ij} \qquad \hat{D}_{ij} = \hat{D}(x_i,x_j). \tag{2.4}$$

Here $\hat{D}(x,y)$ is a function of two arguments. The element $\hat{D}_{ij}$ depends on i and j only by means of its arguments x_i and x_j.

First, it is useful to rewrite the representation (2.2) in terms of a determinant of a matrix $n\times n$. To do this, notice that $\sigma_{2n-j} = 0$, if $j < n$, so $M_{nj} = \delta_{nj}[s]\prod_{m=1}^{n} x_m$. Thus, we obtain

$$Q_n(s) = [s]^{-1} \prod_{m=1}^{n} x_m^{-1} \det_n M_{ij}(s) \qquad i,j = 1,\ldots,n. \tag{2.5}$$

The right-hand side of (2.5) is well defined for all $s \neq 0$ and $n \neq 0$. These two cases should be considered separately. It is easy to see that for $s = 0$ and $n \neq 0$ one has to understand (2.5) as a limit $s \to 0$, because the determinant is proportional to $[s]$. However, for $n = 0$, the original representation (2.2) is not well defined, while it is natural to define the determinant $\det_0 M_{ij} = 1$ in (2.5). So, we obtain $Q_0(s) = [s]^{-1}$ for $s \neq 0$. On the other hand, the case $s = 0$ corresponds to the form factor of the local field. In this case we have $H_0 = 0$, and the form factor is equal to zero, $H_0 Q_0(0) = F_0 = 0$. Thus, we define $Q_0(s) = [s]^{-1}$ for $s \neq 0$. We do not define $Q_0(0)$, but we simply put $F_0 = 0$ for $s = 0$. Our definition of Q_0 leads to the formula for the vacuum expectation value $F_0 = H_0/[s]$. We have checked that this is consistent with [23].

In order to study correlation functions we need to find the square of the polynomials $Q_n(s)$,

$$Q_n^2(s) = [s]^{-2} \prod_{m=1}^{n} x_m^{-2} \det{}_n C_{jk} \tag{2.6}$$

where

$$C_{jk} = (M^{\mathrm{T}} \cdot M)_{jk} = \sum_{i=1}^{n} [i - j + s][i - k + s] \sigma_{2i-j} \sigma_{2i-k}. \tag{2.7}$$

One can calculate the sum in (2.7) using an integral representation for elementary symmetric polynomials

$$\sigma_k = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}z}{z^{n-k+1}} \prod_{m=1}^{n} (z + x_m). \tag{2.8}$$

Here the integral is taken in the positive direction with respect to an arbitrary circle $|z| = \rho$ around the origin. Choosing the radius of the circle $\rho > 1$ and using (A.5) from appendix A, we find

$$\begin{aligned} C_{jk} = \frac{1}{(2\pi \mathrm{i})^2} \oint \mathrm{d}^2 z \frac{\prod_{m=1}^{n} (z_1 + x_m)(z_2 + x_m)}{(q - q^{-1})^2} z_1^{n-j+1} z_2^{n-k+1} \\ \times \left\{ \frac{q^{2n+2+2s-j-k}}{q^2 z_1^2 z_2^2 - 1} + \frac{q^{-2n-2-2s+j+k}}{q^{-2} z_1^2 z_2^2 - 1} - \frac{q^{k-j}}{z_1^2 z_2^2 - 1} - \frac{q^{j-k}}{z_1^2 z_2^2 - 1} \right\}. \end{aligned} \tag{2.9}$$

In order to find a common denominator we make replacements of variables in the braces: $z_1 q^{1/2} = w_1$, $z_2 q^{1/2} = w_2$ in the first term; $z_1 q^{-1/2} = w_1$, $z_2 q^{-1/2} = w_2$ in the second term; $z_1 q^{1/2} = w_1$, $z_2 q^{-1/2} = w_2$ in the third term and $z_1 q^{-1/2} = w_1$, $z_2 q^{1/2} = w_2$ in the fourth term. After simple algebra we arrive at

$$C_{jk} = \frac{1}{(2\pi \mathrm{i})^2} \oint \mathrm{d}^2 w \frac{w_1^{n-j+1} w_2^{n-k+1}}{w_1^2 w_2^2 - 1} G^{(j)}(w_1) G^{(k)}(w_2) \tag{2.10}$$

where

$$G^{(\ell)}(w) = \frac{1}{q - q^{-1}} \left(q^{s+(n-\ell)/2} \prod_{m=1}^{n} (w q^{-1/2} + x_m) - q^{-s-(n-\ell)/2} \prod_{m=1}^{n} (w q^{1/2} + x_m) \right). \tag{2.11}$$

The matrix C still depends on n^2 different functions C_{jk}. However, this matrix can be transformed to a more convenient form. Let us introduce the matrix A_{jk} (it is studied in appendix B)

$$A_{jk} = \frac{1}{(n-j)!} \frac{\mathrm{d}^{n-j}}{\mathrm{d}x^{n-j}} \prod_{m \neq k}^{n} (x + x_m)|_{x=0} \tag{2.12}$$

with a determinant

$$\det A = \prod_{a<b}^{n}(x_a - x_b). \tag{2.13}$$

Instead of matrix C it will be convenient to introduce matrix D

$$D = A^{\mathrm{T}} C A. \tag{2.14}$$

Determinants of matrices C and D are related by

$$\det_n C = \prod_{a>b}^{n}(x_a - x_b)^{-2} \det_n D. \tag{2.15}$$

The calculation of the explicit expression for matrix D in (2.14) reduces to the summation of the Taylor series (see (B.8)), so we have

$$D_{jk} = \oint \mathrm{d}^2 w \frac{w_1 w_2}{(2\pi \mathrm{i})^2 (w_1^2 w_2^2 - 1)} Y(w_1, x_j) Y(w_2, x_k) \tag{2.16}$$

where

$$Y(w, x) = \frac{J(w)}{q - q^{-1}} \left(\frac{q^s}{wq^{1/2} + x} - \frac{q^{-s}}{wq^{-1/2} + x} \right) \tag{2.17}$$

and

$$J(w) = \prod_{m=1}^{n}(wq^{1/2} + x_m)(wq^{-1/2} + x_m). \tag{2.18}$$

Taking the integral, for instance, with respect to w_2 (recall that $|w_1 w_2| > 1$), we have after the symmetrization of the integrand

$$D_{jk} = \frac{1}{8\pi \mathrm{i}} \oint \frac{\mathrm{d}w}{w} (Y(w, x_j) + Y(-w, x_j))(Y(w^{-1}, x_k) + Y(-w^{-1}, x_k)). \tag{2.19}$$

Thus, we obtain a new representation for the square of the polynomial $Q_n(s)$:

$$Q_n^2(s) = \frac{\det_n D}{[s]^2 \prod_{m=1}^{n} x_m^2 \prod_{a>b}^{n} (x_a - x_b)^2}. \tag{2.20}$$

This brings us closer to (2.1).

3. Dual fields

The entries of the matrix D_{jk} are parametrized now by a single function D (2.19). However, an element D_{jk}, is still not a function of two arguments only, because of the product $J(w) = \prod_{m=1}^{n}(wq^{1/2} + x_m)(wq^{-1/2} + x_m)$. This product depends on all x_m. In order to get rid of these products we introduce an auxiliary Fock space and auxiliary quantum operators—dual fields. Dual fields are linear combinations of canonical Bose fields, see [6, p 210].

Let us define

$$\Phi_1(x) = q_1(x) + p_2(x) \qquad \Phi_2(x) = q_2(x) + p_1(x) \tag{3.1}$$

where the operators $p_j(x)$ and $q_j(x)$ act in the canonical Bose Fock space in the following way:

$$(0|q_j(x) = 0 \qquad p_j(x)|0) = 0. \tag{3.2}$$

Non-zero commutation relations are given by

$$[p_1(x), q_1(y)] = [p_2(x), q_2(y)] = \xi(x, y) = \log((x + yq^{1/2})(x + yq^{-1/2})). \tag{3.3}$$

Due to the symmetry of the function $\xi(x, y) = \xi(y, x)$, all fields $\Phi_j(x)$ commute with each other

$$[\Phi_j(x), \Phi_k(y)] = 0 \qquad j, k = 1, 2. \tag{3.4}$$

However, despite these simple commutation relations, the vacuum expectation value of the dual fields may be non-trivial, for example

$$(0|\Phi_1(x)\Phi_2(y)|0) = (0|p_2(x)q_2(y)|0) = \xi(x, y). \tag{3.5}$$

It is easy to show that an exponent of the dual field acts like a shift operator. Namely, if $f(\Phi_1(y))$ is a function of $\Phi_1(y)$ then

$$\begin{aligned}(0|\prod_{m=1}^{n} \mathrm{e}^{\Phi_2(x_m)} f(\Phi_1(y))|0) &= (0|\prod_{m=1}^{n} \mathrm{e}^{p_1(x_m)} f(q_1(y))|0) \\ &= (0|f\left(q_1(y) + \sum_{m=1}^{n} \xi(x_m, y)\right)|0) = f(\log J(y)).\end{aligned} \tag{3.6}$$

Using this property of dual fields one can remove the products $J(w)$ from the matrix D_{jk}.

Let us define

$$\hat{Y}(w, x) = \frac{\mathrm{e}^{\Phi_1(w)}}{q - q^{-1}} \left(\frac{q^s}{wq^{1/2} + x} - \frac{q^{-s}}{wq^{-1/2} + x} \right) \tag{3.7}$$

and

$$\hat{D}_{jk} = \frac{1}{8\pi \mathrm{i}} \oint \frac{\mathrm{d}w}{w} (\hat{Y}(w, x_j) + \hat{Y}(-w, x_j))(\hat{Y}(w^{-1}, x_k) + \hat{Y}(-w^{-1}, x_k)). \tag{3.8}$$

Then, due to (3.6), we have

$$\det{}_n D = (0|\prod_{m=1}^{n} \mathrm{e}^{\Phi_2(x_m)} \det{}_n \hat{D}|0) \tag{3.9}$$

or

$$\det{}_n D = (0|\det{}_n(\hat{D}(x_j, x_k)\, \mathrm{e}^{\frac{1}{2}\Phi_2(x_j) + \frac{1}{2}\Phi_2(x_k)})|0). \tag{3.10}$$

The entries of the matrix $\hat{D}_{jk}$ depend on x_j and x_k only, and they do not depend on other variables x_m. Thus, we have presented the square of the polynomial $Q_n(s)$ in terms of a vacuum expectation value of a determinant of a matrix $n \times n$, similar to one of the terms on the right-hand side of (2.1). The entries of matrix D are parametrized by the single two-variable function $\hat{D}(x, y)$. Let us emphasize again that, as an operator in the auxiliary Fock space, $\hat{D}(x, y)$ belongs to an Abelian subalgebra.

Besides the polynomial $Q_n(s)$ the form factor (1.4) is proportional to a double product $\prod_{a>b}^{n} F_{\min}(\beta_{ab})(x_a + x_b)^{-1}$. In order to transform (1.14) to (2.1) it is necessary to factorize this product. To do this we introduce another dual field

$$\tilde{\Phi}_0(x) = \tilde{q}_0(x) + \tilde{p}_0(x). \tag{3.11}$$

As usual

$$(0|\tilde{q}_0(x) = 0 \qquad \tilde{p}_0(x)|0) = 0. \tag{3.12}$$

Operators $\tilde{q}_0(x)$ and $\tilde{p}_0(y)$ commute with all p_j and q_j ($j = 1, 2$). The only non-zero commutation relation is

$$[\tilde{p}_0(x), \tilde{q}_0(y)] = \eta(x, y) \tag{3.13}$$

where

$$\eta(x, y) = \eta(y, x) = 2\log\left|\frac{F_{\min}(\log(x/y))}{x^2 - y^2}\right|. \tag{3.14}$$

Here we have used the fact that $|F_{\min}(z)| = |F_{\min}(-z)|$. It is worth mentioning also that the right-hand side of (3.14) has no singularity at $x = y$, because $F_{\min}(x)$ has the first-order zero at $x = 0$ and $F'_{\min}(0) = (\mathrm{i}\sin(\pi B/2)F_{\min}(\mathrm{i}\pi))^{-1}$ (see [1]). Hence,

$$\eta(x, x) = -2\log\left|2x^2 \sin\frac{\pi B}{2} F_{\min}(\mathrm{i}\pi)\right|. \tag{3.15}$$

Newly introduced dual fields also commute,

$$[\tilde{\Phi}_0(x), \tilde{\Phi}_0(y)] = 0 = [\tilde{\Phi}_0(x), \Phi_j(y)]. \tag{3.16}$$

However, due to the Campbell–Hausdorff formula, we have

$$(0|\prod_{m=1}^{n} \mathrm{e}^{\tilde{\Phi}_0(x_m)}|0) = \prod_{a,b=1}^{n} \mathrm{e}^{\frac{1}{2}\eta(x_a,x_b)} = \lambda^{-n}\prod_{m=1}^{n} x_m^{-2} \prod_{a>b}^{n}\left|\frac{F_{\min}(\log(x_a/x_b))}{x_a^2 - x_b^2}\right|^2 \tag{3.17}$$

where

$$\lambda = \left|2\sin\frac{\pi B}{2} F_{\min}(\mathrm{i}\pi)\right|. \tag{3.18}$$

Combining the last formula and the representations (2.20) and (3.10) for $Q_n^2(s)$, we find

$$Q_n^2(s)\prod_{a>b}^{n}\left|\frac{F_{\min}(\log(x_a/x_b))}{x_a + x_b}\right|^2 = \frac{\lambda^n}{[s]^2}(0|\det_n \hat{V}(x_j, x_k)|0). \tag{3.19}$$

Here

$$\hat{V}(x_j, x_k) = \hat{D}_{jk}\,\mathrm{e}^{\frac{1}{2}\Phi_0(x_j) + \frac{1}{2}\Phi_0(x_k)} \tag{3.20}$$

and

$$\Phi_0(x) = \tilde{\Phi}_0(x) + \Phi_2(x). \tag{3.21}$$

Therefore, we have managed to represent a square of an absolute value of the form factor as a determinant, similar to one of the terms on the right-hand side of (2.1). In the next section we shall sum up all contributions of the form factors and obtain a determinant representation for a correlation function.

4. The determinant representation for a correlation function

In the previous sections we have obtained the representation for a square of an absolute value of the form factor

$$F_n(\beta_1, \ldots, \beta_n) = \langle 0|\mathcal{O}(0, 0)|\beta_1, \ldots, \beta_n\rangle \tag{4.1}$$

in terms of a determinant

$$|F_n(\beta_1, \ldots, \beta_n)|^2 = \frac{1}{[s]^2}|H_n|^2\lambda^n(0|\det_n \hat{V}(x_j, x_k)|0). \tag{4.2}$$

Here, the constants H_n are equal to

$$H_{2n+1} = H_1\mu^n \qquad H_{2n} = H_0\mu^n \qquad \mu = \frac{4\sin(\pi B/2)}{F_{\min}(\mathrm{i}\pi)} \tag{4.3}$$

and

$$\frac{1}{[s]}H_0 = F_0 = \langle 0|\mathcal{O}(0,0)|0\rangle \qquad H_1 = F_1 = \langle 0|\mathcal{O}(0,0)|\beta\rangle. \tag{4.4}$$

We have the following representation for a correlation function of operators $\mathcal{O}$ in terms of form factors:

$$\langle 0|\mathcal{O}(0,0)\mathcal{O}(x,t)|0\rangle = \sum_{n=0}^{\infty}\int \frac{\mathrm{d}^n\beta}{n!(2\pi)^n}|F_n(\beta_1,\ldots,\beta_n)|^2\prod_{j=1}^{n}\mathrm{e}^{-\theta(x_j)} \tag{4.5}$$

where

$$\theta(x) = \frac{mr}{2}(x + x^{-1}). \tag{4.6}$$

Substituting with (4.2) and (4.3) here, we arrive at the following representation:

$$\begin{aligned}\langle 0|\mathcal{O}(0,0)\mathcal{O}(x,t)|0\rangle = (0|\frac{1}{[s]^2}\Bigg\{&\frac{|H_0|^2+|H_1|^2|\mu|^{-1}}{2}\sum_{n=0}^{\infty}\int_0^{\infty}\frac{\mathrm{d}^n x}{n!}\left(\frac{|\lambda\mu|}{2\pi}\right)^n\\ &\times \det_n\left[\frac{\hat V(x_j,x_k)}{\sqrt{x_jx_k}}\mathrm{e}^{-\frac{1}{2}(\theta(x_j)+\theta(x_k))}\right] + \frac{|H_0|^2-|H_1|^2|\mu|^{-1}}{2}\\ &\times\sum_{n=0}^{\infty}\int_0^{\infty}\frac{\mathrm{d}^n x}{n!}\left(-\frac{|\lambda\mu|}{2\pi}\right)^n\det_n\left[\frac{\hat V(x_j,x_k)}{\sqrt{x_jx_k}}\mathrm{e}^{-\frac{1}{2}(\theta(x_j)+\theta(x_k))}\right]\Bigg\}|0).\end{aligned} \tag{4.7}$$

Both of these series have the form (2.1), so they can be summed up and written as determinants of integral operators (Fredholm determinants)

$$\begin{aligned}\langle 0|\mathcal{O}(0,0)\mathcal{O}(x,t)|0\rangle = (0|\frac{1}{[s]^2}\Bigg\{&\frac{|H_0|^2+|H_1|^2|\mu|^{-1}}{2}\det(I+\gamma\hat U)\\ &+\frac{|H_0|^2-|H_1|^2|\mu|^{-1}}{2}\det(I-\gamma\hat U)\Bigg\}|0)\end{aligned} \tag{4.8}$$

where

$$\hat U(x,y) = \frac{\hat V(x,y)}{\sqrt{xy}}\mathrm{e}^{-\frac{1}{2}(\theta(x)+\theta(y))} \tag{4.9}$$

and

$$\gamma = \frac{4}{\pi}\sin^2\frac{\pi B}{2}. \tag{4.10}$$

The determinant representation (4.8) is the main result of this paper, therefore we summarize here the basic definitions.

The integral operators $I \pm \gamma\hat U$ act on a trial function $f(x)$ as

$$[(I\pm\gamma\hat U)f](x) = f(x) \pm \gamma\int_0^{\infty}\hat U(x,y)f(y)\,\mathrm{d}y. \tag{4.11}$$

The kernel $\hat U(x,y)$ is equal to

$$\hat U(x,y) = \frac{\hat D(x,y)}{\sqrt{xy}}\mathrm{e}^{-\frac{1}{2}(\theta(x)+\theta(y))}\mathrm{e}^{\frac{1}{2}(\Phi_0(x)+\Phi_0(y))} \tag{4.12}$$

where

$$\hat D(x,y) = \frac{1}{8\pi\mathrm{i}}\oint\frac{\mathrm{d}w}{w}(\hat Y(w,x)+\hat Y(-w,x))(\hat Y(w^{-1},y)+\hat Y(-w^{-1},y)) \tag{4.13}$$

and

$$\hat{Y}(w,x) = \frac{\mathrm{e}^{\Phi_1(w)}}{q-q^{-1}}\left(\frac{q^s}{wq^{1/2}+x} - \frac{q^{-s}}{wq^{-1/2}+x}\right). \tag{4.14}$$

The dual fields $\Phi_0(x)$ and $\Phi_1(x)$ were defined in section 3 (see (3.1) and (3.11)). The main property of these dual fields is that they commute with each other, so the Fredholm determinants $\det(I \pm \gamma\hat{U})$ are well defined. Certainly $\det(I \pm \gamma\hat{U})$ are operators in auxiliary Fock space, but they belong to the Abelian subalgebra. On the other hand, the vacuum expectation value of these operators is non-trivial. It follows from commutation relations (3.3) and (3.13) that, in order to calculate the vacuum expectation value, one should use the following prescription:

$$(0|\prod_{a=1}^{M_1}\mathrm{e}^{\Phi_0(x_a)}\prod_{b=1}^{M_2}\mathrm{e}^{\Phi_1(x_b)}|0) = \prod_{a=1}^{M_1}\prod_{b=1}^{M_1}\mathrm{e}^{\frac{1}{2}\eta(x_a,x_b)}\prod_{a=1}^{M_1}\prod_{b=1}^{M_2}\mathrm{e}^{\xi(x_a,x_b)}. \tag{4.15}$$

Here

$$\eta(x,y) = 2\log\left|\frac{F_{\min}(\log(x/y))}{x^2-y^2}\right| \tag{4.16}$$

and

$$\xi(x,y) = \log((x+yq^{1/2})(x+yq^{-1/2})). \tag{4.17}$$

Recall also that the determinant representation (4.8) is valid for an arbitrary s. If $s=0$, one should understand the right-hand side as a limit $s\to 0$, taking into account the fact that $H_0 = 0$.

Similar Fredholm determinant representations have been useful not only in the theory of correlation functions [4–13], but also in matrix models [14–17]. Work on a determinant representation for correlation functions led to the discovery of a determinant formula for a partition function of the six-vertex model with domain wall boundary conditions [30]. In [31] it was shown that this partition function satisfies the Hirota equation. In [32] it was shown that the determinant formula for the partition function of the six-vertex model can help to solve a long-standing mathematical problem—to prove the alternating sign matrix conjecture.

5. Large r asymptotic

In this section we shall demonstrate how one can find a long-distance asymptotic of a correlation function starting from the Fredholm determinant. We shall reproduce some known results.

The kernel of the integral operator $\hat{U}(x,y)$ can be written in the form

$$\hat{U}(x,y) = \oint \mathrm{d}w\, P_1(w,x)P_2(w,y) \tag{5.1}$$

where projectors P are

$$P_1(w,x) = \frac{1}{8\pi\mathrm{i}w\sqrt{x}}(\hat{Y}(w,x)+\hat{Y}(-w,x))\,\mathrm{e}^{\frac{1}{2}\Phi_0(x)-\frac{1}{2}\theta(x)} \tag{5.2}$$

$$P_2(w,y) = \frac{1}{\sqrt{y}}(\hat{Y}(w^{-1},y)+\hat{Y}(-w^{-1},y))\,\mathrm{e}^{\frac{1}{2}\Phi_0(y)-\frac{1}{2}\theta(y)}. \tag{5.3}$$

Let us recall here that w integration goes along a large contour around zero in the positive direction. A radius of the contour should be greater than 1. The Fredholm determinants of the kernels of type (5.1) can be written as determinants of operators acting in the space of variables 'w'

$$\det(I \pm \gamma \hat{U}(x, y)) = \det(I \pm \gamma \tilde{U}(w_1, w_2)) \tag{5.4}$$

where

$$\tilde{U}(w_1, w_2) = \int_0^\infty \mathrm{d}x\, P_1(w_1, x) P_2(w_2, x). \tag{5.5}$$

The integral operator $\tilde{U}(w_1, w_2)$ acts on a trial function $f(w)$ as

$$[(I + \tilde{U}) f](w_1) = f(w_1) + \oint \tilde{U}(w_1, w_2) f(w_2)\, \mathrm{d}w_2. \tag{5.6}$$

Consider the case $r \to \infty$. Then the value of the integral in (5.5) can be estimated by means of a steepest descent method. The saddle point of the function $\theta(x)$ is $x = x_0 = 1$. Examination of the commutation relations of dual fields (4.15) shows that dual fields can be considered as analytic functions in the vicinity of the real axis. Hence, we can estimate the integral in (5.5) as

$$\tilde{U}(w_1, w_2) = P_1(w_1, 1) P_2(w_2, 1) \left(\sqrt{\frac{2\pi}{mr}} + \mathcal{O}(r^{-3/2}) \right). \tag{5.7}$$

Thus, for the large r asymptotic the kernel $\tilde{U}(w_1, w_2)$ becomes a one-dimensional projector, and its Fredholm determinant is equal to

$$\det(I \pm \gamma \tilde{U}) \to 1 \pm \gamma \oint \mathrm{d}w\, \tilde{U}(w, w). \tag{5.8}$$

In order to calculate a vacuum expectation value of $\tilde{U}(w, w)$ one can use prescription (4.15); however, it is better to write down the dual field $\Phi_0(x)$ in terms of the original fields $\Phi_0(x) = \tilde{\Phi}_0(x) + \Phi_2(x)$. Then the contribution of the fields $\tilde{\Phi}_0(1)$ gives

$$(0|\mathrm{e}^{\tilde{\Phi}_0(1)}|0) = \mathrm{e}^{\frac{1}{2}\eta(1,1)} = \lambda^{-1}. \tag{5.9}$$

To find a contribution of the fields $\Phi_1(w)$ and $\Phi_2(x)$ we can use (3.6) and (2.18)

$$(0|\tilde{U}(w, w)|0) = \frac{\mathrm{e}^{-mr}}{8\pi \mathrm{i}\lambda w} \sqrt{\frac{2\pi}{mr}} (Y_1(w, 1) + Y_1(-w, 1))(Y_1(w^{-1}, 1) + Y_1(-w^{-1}, 1)) \tag{5.10}$$

where

$$Y_1(w, 1) = \frac{(wq^{1/2} + 1)(wq^{-1/2} + 1)}{q - q^{-1}} \left(\frac{q^s}{wq^{1/2} + 1} - \frac{q^{-s}}{wq^{-1/2} + 1} \right) = [s] + [s - 1/2]w. \tag{5.11}$$

After substituting this into (5.8) it becomes clear that only a pole at $w = 0$ contributes into the integral, so we arrive at

$$(0| \oint \mathrm{d}w\, \tilde{U}(w, w)|0) = [s]^2 \lambda^{-1} \sqrt{\frac{2\pi}{mr}}\, \mathrm{e}^{-mr} \tag{5.12}$$

and hence

$$(0| \det(I \pm \gamma \tilde{U})|0) \to 1 \pm [s]^2 \frac{\gamma}{\lambda} \sqrt{\frac{2\pi}{mr}}\, \mathrm{e}^{-mr}. \tag{5.13}$$

Finally, substituting this into (4.8) and using explicit expressions for λ (3.18), μ (4.3) and γ (4.10), we obtain the correct asymptotical expression

$$\langle 0|\mathcal{O}(0, 0)\mathcal{O}(x, t)|0\rangle \to \frac{|H_0|^2}{[s^2]} + |H_1|^2 (2\pi mr)^{-1/2}\, \mathrm{e}^{-mr}. \tag{5.14}$$

Recall that for the correlation function of local fields one should put $H_0 = 0 = H_0/[s]$; therefore, we see that the asymptotic formula (5.14) is well defined for arbitrary s. If $H_1 = 0$ (for the stress-energy tensor), then (5.14) gives a constant for an asymptotic. However, in this case one has to estimate the kernel $\tilde{U}(w_1, w_2)$ more accurately. Namely, one should take into account the explicit expression for corrections of order $r^{-3/2}$ in (5.7). In this case the kernel $\tilde{U}$ turns into a two-dimensional projector and it is easy to show that the exponentially decreasing term behaves like $\exp(-2mr)$.

6. Summary

We were able to sum up contributions of all the form factors and obtain the closed expression for correlation functions (4.8). In a following paper we shall use the determinant representation for correlation functions in order to calculate the asymptotics of the time- and temperature-dependent correlation functions. It is clear from [6] how to deform the determinant representation in order to include the temperature dependence. It is also clear how to use this determinant representation in order to evaluate the asymptotics of the time- and temperature-dependent correlation functions of the model [9]. This will provide important information about the physics of the model.

We believe that the determinant representation for correlation functions is a common feature of integrable models.

Acknowledgments

We are grateful to A Berkovich, G Mussardo, F Lessage, S Lukyanov and T Oota for useful discussions. This work was supported in part by NSF under grant No PHY-9605226 and by the RFBR under grant No 96-01-00344.

Appendix A. Elementary symmetric polynomials

Consider an integral representation (2.8) for the elementary symmetric polynomials

$$\sigma_k^{(n)}(x_1, \ldots, x_n) \equiv \sigma_k = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=\rho} \frac{\mathrm{d}z}{z^{n-k+1}} \prod_{m=1}^{n} (z + x_m) \tag{A.1}$$

where ρ is arbitrary positive. Notice that the representation (A.1) holds for an arbitrary integer k including $k < 0$ and $k > n$.

Here we derive an auxiliary formula, which is used in section 2. Namely, let us consider the sum

$$T_{jk}(\alpha) = \sum_{i=1}^{n} \alpha^{2i} \sigma_{2i-j} \sigma_{2i-k} \qquad j, k = 1, \ldots, n \tag{A.2}$$

where α is an arbitrary complex number. One can extend the summation in (A.2) from $-\infty$ to n. Then we have

$$T_{jk}(\alpha) = \sum_{i=-\infty}^{n} \alpha^{2i} \sigma_{2i-j} \sigma_{2i-k} = \sum_{l=0}^{\infty} \alpha^{2(n-l)} \sigma_{2n-2l-j} \sigma_{2n-2l-k}. \tag{A.3}$$

Using the integral representation (A.1) we find

$$T_{jk}(\alpha) = \frac{1}{(2\pi \mathrm{i})^2} \sum_{l=0}^{\infty} \oint \frac{\mathrm{d}^2 z \alpha^{2n-2l}}{z_1^{2l+j-n+1} z_2^{2l+k-n+1}} \prod_{m=1}^{n} (z_1 + x_m)(z_2 + x_m). \tag{A.4}$$

We can choose the integration contour in such a way that $|\alpha z_1 z_2| > 1$ at the contour. Then one can sum up the series with respect to l:

$$T_{jk}(\alpha) = \frac{\alpha^{2n+2}}{(2\pi \mathrm{i})^2} \oint \mathrm{d}^2 z \frac{z_1^{n-j+1} z_2^{n-k+1}}{\alpha^2 z_1^2 z_2^2 - 1} \prod_{m=1}^{n} (z_1 + x_m)(z_2 + x_m). \tag{A.5}$$

The integrand contains only two simple poles $\alpha z_1 z_2 = \pm 1$; therefore, one can take the integral with respect to z_1 or z_2 and obtain a single integral expression for T_{jk}.

Appendix B. Properties of the Vandermonde matrix

Consider a Vandermonde matrix W_{jk},

$$W_{jk} = z_j^{k-1} \qquad j, k = 1, \ldots, n \tag{B.1}$$

with

$$\det_n(W_{jk}) = \prod_{a>b}^{n} (z_a - z_b). \tag{B.2}$$

The inverse matrix W^{-1} can be written in the form

$$(W^{-1})_{jk} = \frac{1}{(j-1)!} \frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}} \prod_{m \neq k}^{n} \left(\frac{x - z_m}{z_k - z_m} \right) \Bigg|_{x=0}. \tag{B.3}$$

Indeed,

$$\sum_{l=1}^{n} W_{jl} (W^{-1})_{lk} = \sum_{l=0}^{n-1} \frac{z_j^l}{l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l} \prod_{m \neq k}^{n} \left(\frac{x - z_m}{z_k - z_m} \right) \Bigg|_{x=0} = \prod_{m \neq k}^{n} \left(\frac{z_j - z_m}{z_k - z_m} \right) = \delta_{jk}. \tag{B.4}$$

Here we have used the fact that the right-hand side of (B.4) is a Taylor series for the polynomial of the $(n-1)$ degree $\prod_{m \neq k}^{n} (x - z_m)(z_k - z_m)^{-1}$.

In section 2 we used the matrix A_{jk}:

$$A_{jk} = \frac{1}{(n-j)!} \frac{\mathrm{d}^{n-j}}{\mathrm{d}x^{n-j}} \prod_{m \neq k}^{n} (x + x_m)|_{x=0}. \tag{B.5}$$

The determinant of this matrix is equal to

$$\det_n A = \prod_{a \neq b}^{n} (x_a - x_b) \det \left[A_{jk} \prod_{m \neq k} (x_m - x_k)^{-1} \right]. \tag{B.6}$$

It is easy to see that the matrix on the right-hand side of (B.6) coincides with the inverse Vandermonde matrix W^{-1} up to the replacement $x_m = -z_m$ and a permutation of rows. Thus, we obtain

$$\det_n A = \prod_{a<b}^{n} (x_a - x_b). \tag{B.7}$$

The calculation of products of matrix A and matrices containing powers of some complex numbers w is simple. For example, deriving (2.16) we used

$$\begin{aligned} &\sum_{l=1}^{n} \frac{1}{(n-l)!} \frac{\mathrm{d}^{n-l}}{\mathrm{d}x^{n-l}} \prod_{m \neq j}^{n} (x + x_m)|_{x=0} \cdot w_1^{n-l} G^{(l)}(w_1) \\ &\qquad = \frac{q^s}{q - q^{-1}} \prod_{m=1}^{n} (w_1 q^{-1/2} + x_m) \sum_{l=1}^{n} \frac{(w_1 q^{1/2})^{n-l}}{(n-l)!} \frac{\mathrm{d}^{n-l}}{\mathrm{d}x^{n-l}} \prod_{m \neq j}^{n} (x + x_m)|_{x=0} \end{aligned}$$

$$
-\frac{q^{-s}}{q-q^{-1}}\prod_{m=1}^{n}(w_1q^{1/2}+x_m)\sum_{l=1}^{n}\frac{(w_1q^{-1/2})^{n-l}}{(n-l)!}\frac{\mathrm{d}^{n-l}}{\mathrm{d}x^{n-l}}\prod_{m\neq j}^{n}(x+x_m)|_{x=0}
$$
$$
=\frac{\prod_{m=1}^{n}(w_1q^{1/2}+x_m)(w_1q^{-1/2}+x_m)}{q-q^{-1}}\left[\frac{q^s}{w_1q^{1/2}+x_j}-\frac{q^{-s}}{w_1q^{-1/2}+x_j}\right]. \tag{B.8}
$$

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